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Nakano Semipositivity of the direct images of pluricanonical systems

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1 Introduction

In this paper, I would like to explain my recent works on the direct image of pluricanonical system and adjoint line bundles ([T5, T6]).

1.1 Semipositivity theorem

Our starting point is the following theorem.

Theorem 1.1 ([Ka1, p.57, Theorem 1]) *Let $f : X \rightarrow C$ be an algebraic fiber space over a projective curve C . Then $F_m := f_* \mathcal{O}_X(mK_{X/S})$ is a semipositive vector bundle on S in the sense that for any quotient sheaf \mathcal{Q} of $f_* \mathcal{O}_X(mK_{X/S})$, $\deg_C \mathcal{Q} \geq 0$ holds. \square*

Theorem 1.1 has been used in many contexts in algebraic geometry ([Ka1, Ka2, V1, V2]). The original proof of Theorem 1.1 is based on the fact that the hermitian metric

$$\|\eta\|_{\frac{1}{m}} := \left(\int_{X_s} (\eta \wedge \bar{\eta})^{\frac{1}{m}} \right)^{\frac{m}{2}}, \eta \in H^0(X_s, \mathcal{O}_{X_s}(mK_{X_s}))$$

on the tautological line bundle on $\mathbb{P}(F_m^*)$ has semipositive curvature. It is natural to consider the following problem.

Problem 1.2 *Does F_m admits a natural hermitian metric with semipositive curvature? \square*

The purpose of this paper is to show that F_m has a natural continuous metric with semipositive curvature in the sense of Nakano:

Theorem 1.3 ([T5]) *Let $f : X \rightarrow S$ be projective family such that X and S are smooth. Let S° be a nonempty Zariski open subset such that f is smooth over S° . Then $K_{X/S}$ has a relative AZD h over S° such that Θ_h is semipositive on X .*

And $F_m := f_ \mathcal{O}_X(mK_{X/S})$ carries a continuous hermitian metric h_{F_m} with Nakano semipositive curvature in the sense of current over S° .*

Let $x \in S - S^\circ$ be a point and let σ be a local holomorphic section of F_m on a neighbourhood U of x . Then $\sqrt{-1} \partial \bar{\partial} \log h_{F_m}(\sigma, \sigma)$ extends as a closed positive current across $(S - S^\circ) \cap U$. \square

By the L^2 -extension theorem ([O, O-T]), Theorem 1.3 immediately implies the following theorem.

Corollary 1.4 ([T3]) *Let $f : X \rightarrow S$ be a smooth projective family. Then $P_m(X_s) = \dim H^0(X_s, \mathcal{O}_{X_s}(mK_{X_s}))$ is independent of $s \in S$. \square*

1.2 Variation of Bergman kernels

One of the main tool of the proof of Theorem 1.3 is a generalization of the recent results of Berndtsson ([B1, B2]).

Theorem 1.5 ([B1]) *Let D be a pseudoconvex domain in $\mathbb{C}_z^n \times \mathbb{C}_t^k$. And let ϕ be a plurisubharmonic function on D . For $t \in \Delta$, we set $D_t := \Omega \cap (\mathbb{C}^n \times \{t\})$ and $\phi_t := \phi|_{D_t}$. Let $K(z, t) (t \in \mathbb{C}_t^k)$ be the Bergman kernel of the Hilbert space*

$$A^2(D_t, e^{-\phi_t}) := \{f \in \mathcal{O}(\Omega_t) \mid \int_{D_t} e^{-\phi_t} |f|^2 < +\infty\}.$$

Then $\log K(z, t)$ is a plurisubharmonic function on D . \square

This is a generalization of the former result of Maitani and Yamaguchi ([M-Y]). As in mentioned in [B2], his proof also works for a pseudoconvex domain in a locally trivial family of manifolds which admits a Zariski dense Stein subdomain.

He also prove the following positivity theorem.

Theorem 1.6 ([B2, Theorem 1.1]) *Let us consider a domain $D = U \times \Omega$ and let ϕ be a plurisubharmonic function on D . For simplicity we assume that ϕ is smooth up to the boundary and strictly plurisubharmonic in D . Then for each $t \in U$, $\phi_t := \phi(\cdot, t)$ is plurisubharmonic on Ω . Let A_t^2 be the Bergman space of holomorphic functions on Ω with norm*

$$\|f\|^2 = \|f\|_t^2 := \int_{\Omega} e^{-\phi_t} |f|^2.$$

The spaces A_t^2 are all equal as vector spaces but have norms that vary with t . Then "infinite rank" vector bundle E over U with fiber $E_t = A_t^2$ is therefore trivial as a bundle but is equipped with a nontrivial metric. Then $(E, \|\cdot\|_t)$ is strictly positive in the sense of Nakano. \square

In Theorem 1.5 the assumption that D is a pseudoconvex domain in the product space is rather strong. And in Theorem 1.6, Berndtsson also assumed that D is a product. Our first aim is to remove these assumptions and generalize Theorems 1.5, 1.6 to the case of adjoint line bundles smooth projective fibrations.

By using this generalization we can study non locally trivial algebraic fiber space.

To state our theorem, let us introduce the notion of the Bergman kernels of adjoint line bundles. Let X be a complex manifold of dimension n and let (L, h) be a singular hermitian line bundle on X . Let K_X denote the canonical line bundle on X . Let $A^2(X, K_X + L, h)$ be the Hilbert space defined by

$$A^2(X, K_X + L, h) := \{\sigma \in H^0(X, \mathcal{O}_X(K_X + L)) \mid (\sqrt{-1})^{n^2} \int_X h \cdot \sigma \wedge \bar{\sigma} < +\infty\},$$

where we have defined the inner product on $A^2(X, K_X + L, h)$ by

$$(\sigma, \tau) := (\sqrt{-1})^{n^2} \int_X h \cdot \sigma \wedge \bar{\tau}.$$

We define the Bergman kernel $K(X, K_X + L, h)$ of the adjoint bundle $K_X + L$ with respect to h by

$$K(X, K_X + L, h) := (\sqrt{-1})^{n^2} \sum_i \sigma_i \wedge \bar{\sigma}_i.$$

where $\{\sigma_i\}$ is a complete orthonormal basis of the Hilbert space $A^2(X, K_X + L, h)$. Then $K(X, K_X + L, h)$ is independent of the choice of the complete orthonormal basis $\{\sigma_i\}$. In fact

$$K(X, K_X + L, h)(x) = \sup\{(\sqrt{-1})^{n^2} \sigma(x) \wedge \bar{\sigma}(x) \mid \|\sigma\| = 1\}$$

holds.

Now we shall state our theorem.

Theorem 1.7 ([T5]) *Let $f : X \rightarrow S$ be a smooth projective family of projective varieties over a complex manifold S . Let (L, h) be a singular hermitian line bundle on X such that Θ_h is semipositive on X . Let $K_s := K(X_s, K_X + L|_{X_s}, h|_{X_s})$ be the Bergman kernel of $K_X + (L|_{X_s})$ with respect to $h|_{X_s}$. Then the singular hermitian metric h_B of $K_{X/S} + L$ defined by*

$$h_B|_{X_s} := K_s^{-1}$$

has semipositive curvature on X . \square

Theorem 1.7 follows from Theorem 1.5 by a simple trick as follows. We may assume that S is the unit open disk Δ centered at O . $f : X \rightarrow S$ is not locally trivial. We shall embed X into the trivial family $p : X \times \Delta \rightarrow \Delta, p(x, t) = x (x \in X, t \in \Delta)$ by

$$i : X \rightarrow X \times \Delta$$

defined by

$$i(x) := (x, f(x)).$$

Then $i(X)$ is a hypersurface in $X \times \Delta$ and not a domain in $X \times \Delta$. So we shall thicken $i(X)$ by replacing $X_t (t \in \Delta)$ by $f^{-1}(\Delta(t, \varepsilon))$, where $\Delta(t, \varepsilon)$ denotes the open disk of radius ε centered at t . In this way we construct a thickened family

$$f_\varepsilon : X(\varepsilon) \rightarrow \Delta(1/2)$$

which is considered to be a pseudoconvex domain in the product family $X \times \Delta(1/2)$ over $\Delta(1/2)$, where $\Delta(1/2)$ denotes $\Delta(0, 1/2)$. Then Theorem 1.5 is applicable to the family of Bergman kernels of the adjoint bundle of $p^*(L, h)$ over $\Delta(1/2)$. Letting ε tend to 0, with the rescaling constant $\pi\varepsilon^2$, we obtain Theorem 1.7.

By entirely the same method, we also generalize Theorem 1.6 as follows.

Theorem 1.8 *Let $f : X \rightarrow S$ be a smooth projective family of over a complex curve S of relative dimension n . Let (L, h) be a hermitian line bundle on X*

such that Θ_h is semipositive on X . We define the hermitian metric h_E on $E := f_*\mathcal{O}_X(K_{X/S} + L)$ by

$$h_E(\sigma, \tau) := (\sqrt{-1})^{n^2} \int_{X_s} h \cdot \sigma \wedge \bar{\tau}.$$

Then (E, h_E) is semipositive in the sense of Nakano. Moreover if Θ_h is strictly positive, then (E, h_E) is strictly positive in the sense of Nakano. \square

After I completed this work, I have received a preprint of Berndtsson [B3], which proved Theorem 1.7 under the assumption that h is C^∞ . His proof is more computational than the one in [T5] and it is not clear whether his proof works also for a singular h .

The proof in [T5] is very simple and based on the original proof of Theorem 1.5 in [B1].

2 Preliminaries

Definition 2.1 L is said to be **pseudoeffective**, if there exists a singular hermitian metric h on L such that the curvature current Θ_h is a closed positive current. Also a singular hermitian line bundle (L, h) is said to be **pseudoeffective**, if the curvature current Θ_h is a closed positive current. \square

Here we shall introduce the notion of analytic Zariski decompositions. By using analytic Zariski decompositions, we can handle big line bundles like nef and big line bundles.

Definition 2.2 Let M be a compact complex manifold and let L be a holomorphic line bundle on M . A singular hermitian metric h on L is said to be an **analytic Zariski decomposition**, if the followings hold.

1. Θ_h is a closed positive current,
2. for every $m \geq 0$, the natural inclusion

$$H^0(M, \mathcal{O}_M(mL) \otimes \mathcal{I}(h^m)) \rightarrow H^0(M, \mathcal{O}_M(mL))$$

is an isomorphism. \square

Remark 2.3 If an AZD exists on a line bundle L on a smooth projective variety M , L is pseudoeffective by the condition 1 above. \square

Theorem 2.4 ([T1, T2]) Let L be a big line bundle on a smooth projective variety M . Then L has an AZD.

As for the existence for general pseudoeffective line bundles, now we have the following theorem.

Theorem 2.5 ([D-P-S, Theorem 1.5]) Let X be a smooth projective variety and let L be a pseudoeffective line bundle on X . Then L has an AZD.

Theorem 2.6 ([O, Theorem 4]) *Let M be a complex manifold with a continuous volume form dV_M , let E be a holomorphic vector bundle over M with C^∞ -fiber metric h_E , let S be a closed complex submanifold of M , let $\Psi \in \mathfrak{H}(S)$ and let K_M be the canonical bundle of M . Then $(S, dV_M(\Psi))$ is a set of interpolation for $(E \otimes K_M, h_E \otimes (dV_M)^{-1}, dV_M)$, if the followings are satisfied.*

1. *There exists a closed set $X \subset M$ such that*
 - (a) *X is locally negligible with respect to L^2 -holomorphic functions, i.e., for any local coordinate neighbourhood $U \subset M$ and for any L^2 -holomorphic function f on $U \setminus X$, there exists a holomorphic function \tilde{f} on U such that $\tilde{f}|_{U \setminus X} = f$.*
 - (b) *$M \setminus X$ is a Stein manifold which intersects with every component of S .*
2. $\Theta_{h_E} \geq 0$ *in the sense of Nakano,*
3. $\Psi \in \mathfrak{H}(S) \cap C^\infty(M \setminus S)$,
4. $e^{-(1+\epsilon)\Psi} \cdot h_E$ *has semipositive curvature in the sense of Nakano for every $\epsilon \in [0, \delta]$ for some $\delta > 0$.*

Under these conditions, there exists a constant C and an interpolation operator from $A^2(S, E \otimes K_M|_S, h \otimes (dV_M)^{-1}|_S, dV_M[\Psi])$ to $A^2(M, E \otimes K_M, h \otimes (dV_M)^{-1} \cdot dV_M)$ whose norm does not exceed $C\delta^{-3/2}$. If Ψ is plurisubharmonic, the interpolation operator can be chosen so that its norm is less than $2^4\pi^{1/2}$. \square

The above theorem can be generalized to the case that (E, h_E) is a singular hermitian line bundle with semipositive curvature current (we call such a singular hermitian line bundle (E, h_E) a **pseudoeffective singular hermitian line bundle**) as was remarked in [O].

Lemma 2.7 *Let $M, S, \Psi, dV_M, dV_M[\Psi], (E, h_E)$ be as in Theorem 2.6. Let (L, h_L) be a pseudoeffective singular hermitian line bundle on M . Then S is a set of interpolation for $(K_M \otimes E \otimes L, dV_M^{-1} \otimes h_E \otimes h_L)$. \square*

3 Proof of Theorems 1.3

3.1 Dynamical construction of AZD

Let X be a smooth projective variety and let K_X be the canonical line bundle of X . Let n denote the dimension of X . We shall assume that K_X is pseudoeffective. Then by Theorem 2.5, K_X admits an AZD h .

Let A be a sufficiently ample line bundle on X such that for every pseudoeffective singular hermitian line bundle (L, h_L) ,

$$\mathcal{O}_X(A + L) \otimes \mathcal{I}(h_L)$$

and

$$\mathcal{O}_X(K_X + A + L) \otimes \mathcal{I}(h_L)$$

are globally generated. This is possible by [S, p. 667, Proposition 1].

Let h_A be a C^∞ hermitian metric on A with strictly positive curvature.

Let B be another ample line bundle on X and let h_B be a C^∞ hermitian metric on B with strictly positive curvature. Let ℓ be an arbitrary positive integer greater than or equal to 2.

We shall construct an AZD of $B + \ell K_X$ as follows.

Let $K(A + B + K_X, h_A \cdot h_B)$ be the Bergman kernel of $A + B + K_X$ with respect to $h_A \cdot h_B$. We define the singular hermitian metric h_1 on $A + B + K_X$ by

$$h_1 := K(A + B + K_X, h_A \cdot h_B)^{-1}.$$

We define the singular hermitian metric on $A + B + 2K_X$ by

$$h_2 := K(A + B + 2K_X, h_1)^{-1}.$$

We continue this process until we obtain the singular hermitian metric h_ℓ on $A + B + \ell K_X$.

Next we define the singular hermitian metric $h_{\ell+1}$ on $A + 2B + (\ell + 1)K_X$ by

$$h_{\ell+1} := K(A + 2B + (\ell + 1)K_X, h_\ell \cdot h_B)^{-1}.$$

And we continue as

$$h_{\ell+2} := K(A + 2B + (\ell + 2)K_X, h_{\ell+1})^{-1}$$

until we obtain $h_{2\ell-1}$.

It is clear that h_m has semipositive curvature in the sense of currents for every $m \geq 1$.

We set

$$K_{m,1/\ell} := 1/h_{m,1/\ell}.$$

Proposition 3.1 (cf. [T3])

$$K_{\infty,1/\ell} := \overline{\lim}_{m \rightarrow \infty} \sqrt[n]{(m!)^{-n} K_{m,1/\ell}}$$

exists and

$$h_{\infty,1/\ell} := 1/K_{\infty,1/\ell}$$

is an AZD of $K_X + \ell^{-1}B$. \square

Proof of Proposition 3.1. There exists a positive constant C such that

$$h^0(X, \mathcal{O}_X(j(B + \ell K_X) + kK_X + A) \otimes \mathcal{I}(h)) \leq C(j\ell + k)^n$$

holds for every $j \geq 1$ and $0 \leq k < \ell$. We set

$$m := j\ell + k.$$

Let dV be a fixed C^∞ volume form on X . Then by the submeanvalue inequality of plurisubharmonic functions, we see that by induction there exists a positive constant C_1 such that for $m = j\ell + k$ ($0 \leq k \leq \ell - 1$)

$$K_{m,1/\ell} \leq C_1^m \cdot (m!)^n dV^m h_A^{-1} h_B^{-j} \quad (1)$$

holds.

Now we shall consider the lower estimate of $K_{m,1/\ell}$. We note that for every $x \in X$ and $m = j\ell + k$,

$$h_m^{-1}(x) = K(A + jB + mK_X, h_{m-1}) =$$

$$\sup\{|\sigma|^2(x); \sigma \in \Gamma(X, \mathcal{O}_X(A + jB + mK_X \otimes \mathcal{I}(h^m))), \int_X h_{m-1} \cdot |\sigma|^2 = 1\}$$

holds by the extremal property of Bergman kernels (cf. [Kr, p.46, Proposition 1.4.16]).

Let h be an AZD of $K_X + \ell^{-1}B$.

Since $(K_X + \ell^{-1}B, h) \mid V$ is big. By Kodaira's lemma, h is dominated by a singular hermitian metric h' such that $\Theta_{h'}$ is strictly positive on V . For $0 < \varepsilon < 1$ we set

$$h_\varepsilon := h^{1-\varepsilon} \cdot (h')^\varepsilon.$$

Then $h < h_\varepsilon$ holds.

Using the L^2 -extension theorem (Theorem 2.6 and Lemma 2.7), we see that there exists a positive constant C_2 such that

$$K_m \geq (m!)^n C_2^m h_A^{-1} h_B^{\frac{k}{\ell}} h_\varepsilon^{-m} \quad (2)$$

holds by induction. Here the factor $(m!)^n$ appears by the fact that Θ_{h_ε} is strictly positive hence we can take local frame e of $\ell K_X + B$ around $x \in V$ and coordinate z_1, \dots, z_ν so that

$$h_\varepsilon(e, e) = (1 - \|z\|^2)h(e, e)(x) + o(\|z\|^2) \quad (3)$$

holds (cf. [Ti, p.105, (1.11)]). and the equality

$$\frac{\sqrt{-1}}{2} \int_{|t|<1} (1 - |t|^2)^m dt \wedge d\bar{t} = \frac{2\pi}{m+1}.$$

By (1) and (2), moving x and letting ε tend to 0, we have that

$$K_{\infty,1/\ell} := \overline{\lim}_{m \rightarrow \infty} \sqrt[n]{(m!)^{-n} K_m}$$

exists and

$$h_{1/\ell} := 1/K_{\infty,1/\ell}$$

is an AZD of $K_X + \ell^{-1}B$.

Now we shall construct an AZD of K_X . We set

$$C(\ell) := \frac{n!}{\ell^n} \overline{\lim}_{m \rightarrow \infty} m^{-n} h^0(X, \mathcal{O}_X(m(\ell K_X + B))).$$

The following proposition follows from (3).

Proposition 3.2

$$K_\infty := \overline{\lim}_{\ell \rightarrow \infty} (C(\ell) \cdot K_{\infty,1/\ell})$$

exists and

$$h_\infty := 1/K_\infty$$

is an AZD of K_X . \square

3.2 Completion of the proof of Theorem 1.3

Let $f : X \rightarrow S$ be a smooth projective morphism. We perform the construction of AZD for K_{X_s} ($s \in S$) simultaneously for all $s \in S$.

Let A be a sufficiently ample line bundle on X with C^∞ hermitian metric h_A with strictly positive curvature. Let B be another ample line bundle on X . Then as in Section 3.1, we construct a family of Bergman kernels $K_{m,1/\ell,s}$ ($s \in S$) of $A|_{X_s} + (jB|_{X_s} + (j\ell + k)K_{X_s})$ ($s \in S, m = j\ell + k$) as in the last subsection.

By Theorem 1.7, we see that if we define $K_{m,1/\ell}$ by

$$K_{m,1/\ell}|_{X_s} = K_{m,1/\ell,s},$$

$$h_{m,1/\ell} = 1/K_{m,1/\ell}$$

is a singular hermitian metric of $A + jB + mK_{X/S}$ with semipositive curvature current by induction on m

Then letting ℓ tend to infinity, we obtain a family of AZD $\{h_{\infty,s}\}$ of K_{X_s} ($s \in S$). If we define a singular hermitian metric h_∞ on $K_{X/S}$ by

$$h_\infty|_{X_s} = h_{\infty,s},$$

then h_∞ is a singular hermitian metric of $K_{X/S}$ with semipositive curvature current. We define a continuous hermitian metric h_{F_m} on $F_m = f_*\mathcal{O}_S(mK_{X/S})$ by

$$h_{F_m}(\sigma, \tau)(s) := (\sqrt{-1})^{n^2} \int_{X_s} h_\infty^{m-1} \sigma \wedge \bar{\tau}$$

Then by Theorem 1.8, we see that h_{F_m} has semipositive curvature in the sense of current.

To complete the proof of Theorem 1.3, we need to consider the asymptotic behavior of h_m around the singular fibers. This can be treated by considering the thickening of fibers, since the thickened fiber is smooth.

4 Applications

As immediate consequences of Theorem 1.3, we obtain simple intrinsic proofs of the following theorems.

Theorem 4.1 ([Sch1]) *Let \mathcal{T}_g be the Teichmüller space of Riemann surfaces of genus g . Let g be the Weil-Petersson metric on \mathcal{T}_g . Then the curvature $\Theta_{g_{WP}}$ of the Kähler metric g_{WP} is strongly negative. \square*

Theorem 4.2 ([V1, V2]) *Let \mathcal{M}_{can} be the moduli space of canonically polarized varieties with only canonical singularities. Then \mathcal{M}_{can} is quasiprojective. \square*

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